

Quasi-morphisms and quasi-states in symplectic topology

Michael Entov*

Abstract. We discuss certain “almost homomorphisms” and “almost linear” functionals that have appeared in symplectic topology and their applications concerning Hamiltonian dynamics, functional-theoretic properties of Poisson brackets and algebraic and metric properties of symplectomorphism groups.

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1. Introduction

Symplectic manifolds carry several interesting mathematical structures of different flavors, coming from algebra, geometry, topology, dynamics and analysis. In this survey we discuss certain “almost homomorphisms” (called *Calabi quasi-morphisms*) on groups of symplectomorphisms of symplectic manifolds and certain “almost linear” functionals (called *symplectic quasi-states*) on the spaces of smooth functions on symplectic manifolds that have been useful in finding new relations between these structures. In particular, we describe applications of these new tools to Hamiltonian dynamics, functional-theoretic properties of Poisson brackets as well as algebraic and metric properties of the groups of symplectomorphisms. We also briefly discuss a relation between the symplectic quasi-states and von Neumann’s mathematical foundations of quantum mechanics. We end the survey with a discussion on the function theory approach to symplectic topology.

A detailed introduction to the subject can be found in the forthcoming book [58] by L.Polterovich and D.Rosen.

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2. Quasi-morphisms and quasi-states - generalities

2.1. Quasi-morphisms. Let G be a group. A function $\mu : G \rightarrow \mathbb{R}$ is called a *quasi-morphism*, if there exists a constant $C > 0$ so that $|\mu(xy) - \mu(x) - \mu(y)| \leq C$ for any $x, y \in G$. We say that a quasi-morphism $\mu : G \rightarrow \mathbb{R}$ is *homogeneous*¹, if $\mu(x^k) = k\mu(x)$ for any $x \in G, k \in \mathbb{Z}$.

Clearly, any \mathbb{R} -valued homomorphism on G is a homogeneous quasi-morphism but finding homogeneous quasi-morphisms that are not homomorphisms is usually a non-trivial task. Let us also note that any homogeneous quasi-morphism μ is conjugacy-invariant and satisfies $\mu(xy) = \mu(x) + \mu(y)$ for any commuting elements x, y (in particular, any homogeneous quasi-morphism on an abelian group is a homomorphism). For more on quasi-morphisms see e.g. [16].

2.2. Quasi-states and quantum mechanics. Roughly speaking, quasi-states are “almost linear” functionals on algebras of a certain kind. The term “quasi-state” comes from the work of Aarnes (see [1] and the references to the Aarnes’ previous work therein) but its history goes back to the mathematical model of quantum mechanics suggested by von Neumann [72]. A basic object of this model is a real Lie algebra of observables that will be denoted by \mathcal{A}_q (q for quantum): its elements (in the simplest version of the theory) are Hermitian operators on a finite-dimensional complex Hilbert space H and the Lie bracket is given by $[A, B]_{\hbar} = \frac{i}{\hbar}(AB - BA)$, where \hbar is the Planck constant. Observables represent physical quantities such as energy, position, momentum etc. In von Neumann’s model a state of a quantum system is given by a functional $\zeta : \mathcal{A}_q \rightarrow \mathbb{R}$ which satisfies the following axioms:

Additivity: $\zeta(A + B) = \zeta(A) + \zeta(B)$ for all $A, B \in \mathcal{A}_q$.

Homogeneity: $\zeta(cA) = c\zeta(A)$ for all $c \in \mathbb{R}$ and $A \in \mathcal{A}_q$.

Positivity: $\zeta(A) \geq 0$ provided $A \geq 0$.

Normalization: $\zeta(Id) = 1$.

As a consequence of these axioms von Neumann proved that for every state ζ there exists a non-negative Hermitian operator U_ζ with trace 1 so that $\zeta(A) = \text{tr}(U_\zeta A)$ for all $A \in \mathcal{A}_q$. An easy consequence of this formula is that for every state ζ there exists an observable A such that

$$\zeta(A^2) - (\zeta(A))^2 > 0. \quad (1)$$

In his book [72] von Neumann adopted a statistical interpretation of quantum mechanics according to which the value $\zeta(A)$ is considered as the expectation of a physical quantity represented by A in the state ζ . In this interpretation the equation (1) says that there are no dispersion-free states. This result led von Neumann to a conclusion which can be roughly described as the impossibility to present random quantum-mechanical phenomena as an observable part of some “hidden”

¹Sometimes homogeneous quasi-morphisms are also called *pseudo-characters* – see e.g. [66].

underlying deterministic mechanism. This conclusion caused a major discussion among physicists (see e.g. [6]) some of whom disagreed with the additivity axiom of a quantum state. Their reasoning was that the formula $\zeta(A+B) = \zeta(A) + \zeta(B)$ makes sense *a priori* only if observables A and B are simultaneously measurable, that is, commute: $[A, B]_{\hbar} = 0$.

In 1957 Gleason [36] proved his famous theorem which can be viewed as an additional argument in favor of von Neumann's additivity axiom. Recall that two Hermitian operators on a finite-dimensional Hilbert space commute if and only if they can be written as polynomials of the same Hermitian operator. Let us define a *quasi-state* on \mathcal{A}_q as an \mathbb{R} -valued functional which satisfies the homogeneity, positivity and normalization axioms above, while the additivity axiom is replaced by one of the two *equivalent* axioms:

Quasi-additivity-I: $\zeta(A+B) = \zeta(A) + \zeta(B)$, provided A and B commute: $[A, B]_{\hbar} = 0$;

Quasi-additivity-II: $\zeta(A+B) = \zeta(A) + \zeta(B)$, provided A and B belong to a singly generated subalgebra of \mathcal{A}_q .

According to the Gleason theorem, every quasi-state ζ on \mathcal{A}_q is linear, that is, a state, provided the complex dimension of the Hilbert space H is at least 3 (it is an easy exercise to show that in the two-dimensional case there are plenty of non-linear quasi-states).

Let us turn now to the mathematical model of classical mechanics. Here the algebra \mathcal{A}_c of observables (c for classical) is the space $C^\infty(M)$ of smooth functions on a symplectic manifold M . The space $C^\infty(M)$ carries two structures. On one hand, it is a Lie algebra with respect to the Poisson bracket (see Section 3.1). On the other hand, it is a dense subset (in the uniform norm) of the commutative Banach algebra $C(M)$ of continuous functions on M . For both frameworks one can define its own version of the notion of a quasi-state adapting, respectively, the first or the second definition of quasi-additivity – as a result one gets the so-called *Lie quasi-states* and *topological quasi-states* (see Section 2.3).

Symplectic quasi-states that appear in symplectic topology and will be discussed below in Section 3 belong to both of these worlds – they are simultaneously Lie and topological quasi-states². Note that for the Lie algebra $C^\infty(M)$ the first definition of quasi-additivity fits in with the physical Correspondence Principle according to which the bracket $[\cdot, \cdot]_{\hbar}$ corresponds to the Poisson bracket $\{\cdot, \cdot\}$ in the classical limit $\hbar \rightarrow 0$. The existence of non-linear symplectic quasi-states on certain symplectic manifolds (see Section 3) can be viewed as an “anti-Gleason phenomenon” in classical mechanics. Interestingly, at least for $M = S^2$, the symplectic quasi-state that we construct is dispersion-free (see Example 3.3), unlike states in von Neumann's model of quantum mechanics. For more information on the connection of symplectic quasi-states to physics see [29] and Remark 4.21 below.

2.3. Lie and topological quasi-states. Here is the precise definition of a Lie quasi-state. Let \mathfrak{g} be a (possibly infinite-dimensional) Lie algebra over \mathbb{R} and

²Interestingly, symplectic quasi-states had appeared in an infinite-dimensional setting in symplectic topology before Lie quasi-states were properly studied in the finite-dimensional setting.

let $W \subset \mathfrak{g}$ be a vector subspace. A function $\zeta : W \rightarrow \mathbb{R}$ is called a *Lie quasi-state*, if it is linear on every abelian subalgebra of \mathfrak{g} contained in W .

Finding non-linear Lie quasi-states is, in general, a non-trivial task: for instance, the difficult Gleason theorem mentioned above is essentially equivalent – in the finite-dimensional setting – to the claim that any Lie quasi-state on the unitary Lie algebra $\mathfrak{u}(n)$, $n \geq 3$, which is bounded on a neighborhood of zero, has to be linear [24]. Choosing an appropriate regularity class of Lie quasi-states is essential for this kind of results: if \mathfrak{g} is finite-dimensional, then any Lie quasi-state on \mathfrak{g} which is differentiable at 0 is automatically linear while the space of all, not necessarily continuous, Lie quasi-states on \mathfrak{g} might be infinite-dimensional [24].

Another source of interest to Lie quasi-states lies in their connection to quasi-morphisms on Lie groups: if \mathfrak{g} is the Lie algebra of a Lie group G and $\mu : G \rightarrow \mathbb{R}$ is a homogeneous quasi-morphism continuous on 1-parametric subgroups, then *the derivative of μ* , that is, the composition of μ with the exponential map, is a Lie quasi-state on \mathfrak{g} , invariant under the adjoint action of G on \mathfrak{g} . Symplectic quasi-states that we will discuss below appear as a particular case of this construction.

Unfortunately, rather little is known about non-linear (continuous) Lie quasi-states and their connections to quasi-morphisms in general – almost all known facts (in particular, a non-trivial description of the space of all non-linear continuous Lie quasi-states on $\mathfrak{sp}(2n, \mathbb{R})$, $n \geq 3$) and some basic open questions on the subject can be found in [24].

Let us now define the notion of a topological quasi-state – it is due to Aarnes [1] (who called it just a “quasi-state”). Let X be a compact Hausdorff topological space and let $C(X)$ be the space of continuous functions on X equipped with the uniform norm. For a function $F \in C(X)$ denote by \mathcal{A}_F the closure in $C(X)$ of the set of functions of the form $p \circ F$, where p is a real polynomial. A functional $\zeta : C(X) \rightarrow \mathbb{R}$ is called a *topological quasi-state* [1], if it satisfies the following axioms:

Quasi-linearity: ζ is linear on \mathcal{A}_F for every $F \in C(X)$ (in particular, ζ is homogeneous).

Monotonicity: $\zeta(F) \leq \zeta(G)$ for $F \leq G$.

Normalization: $\zeta(1) = 1$.

A linear topological quasi-state is called a *state* (similarly to states in von Neumann’s model of quantum mechanics – see Section 2.2). The existence of non-linear topological quasi-states was first proved by Aarnes [1].

By the classical Riesz representation theorem, states on $C(X)$ are in one-to-one correspondence with regular Borel probability measures on X . In [1] Aarnes proved a generalized Riesz representation theorem that associates to each topological quasi-state ζ a *quasi-measure*³ τ on X which is defined only on sets that are either open or closed and is finitely additive but not necessarily sub-additive. The relation between ζ and τ extends the relation between states and measures given by the Riesz representation theorem. In particular, if A is closed, $\tau(A)$ can

³Quasi-measures are sometimes also called *topological measures*.

be thought of as the “value” of ζ on the (discontinuous) characteristic function of A .

3. Calabi quasi-morphisms, symplectic quasi-states

3.1. Symplectic preliminaries. Referring the reader to [47] for the foundations of symplectic geometry we briefly recall the basic notions needed for the further discussion.

Let M^{2n} be a closed connected manifold equipped with a symplectic form ω , that is, a closed and non-degenerate differential 2-form. In terms of classical mechanics, M can be viewed as the phase space of a mechanical system and smooth functions on M (possibly depending smoothly on an additional parameter, viewed as time) are called Hamiltonians. Whenever we consider a time-dependent Hamiltonian we assume that it is 1-periodic in time, i.e. has the form $F : M \times S^1 \rightarrow \mathbb{R}$. Set $F_t := F(\cdot, t)$. The support of F is defined as $\text{supp } F := \cup_{t \in S^1} \text{supp } F_t \subset M$. We say that F is *normalized*, if $\int_M F_t \omega^n = 0$ for any $t \in S^1$.

We denote by $C(M)$ (respectively, $C^\infty(M)$), the space of continuous (respectively, smooth) functions on M and by $\|\cdot\|$ the uniform norm on these spaces: $\|F\| := \max_M |F|$.

Given a (time-dependent) Hamiltonian F , define its *Hamiltonian vector field* X_{F_t} by $\omega(\cdot, X_{F_t}) = dF_t(\cdot)$. Denote the flow of X_{F_t} by ϕ_F^t – it preserves ω and is called the *Hamiltonian flow of F* . Symplectomorphisms of M (that is, diffeomorphisms of M preserving ω) that can be included in such a flow are called *Hamiltonian symplectomorphisms* and form a group $\text{Ham}(M)$ which is a subgroup of the identity component $\text{Symp}_0(M)$ of the full symplectomorphism group $\text{Symp}(M)$. Its universal cover is denoted by $\widetilde{\text{Ham}}(M)$. We say that $\phi_F := \phi_F^1$ is the *Hamiltonian symplectomorphism generated by F* . The Hamiltonian F also generates an element of $\widetilde{\text{Ham}}(M)$ that will be denoted by $\hat{\phi}_F$: it is given by the homotopy class (with the fixed end-points) of the path ϕ_F^t , $0 \leq t \leq 1$, in $\text{Ham}(M)$.

The space of smooth functions on M will be denoted by $C^\infty(M)$. Given $F, G \in C^\infty(M)$, define the *Poisson bracket* $\{F, G\}$ by $\{F, G\} := \omega(X_G, X_F)$. Together with the Poisson bracket $C^\infty(M)$ becomes a Lie algebra whose center is \mathbb{R} (the constant functions).

It is instructive to view $\text{Ham}(M)$ and $\widetilde{\text{Ham}}(M)$ as infinite-dimensional Lie groups whose Lie algebra (the algebra of time-independent Hamiltonian vector fields on M) is naturally isomorphic to $C^\infty(M)/\mathbb{R}$, or to the subalgebra of $C^\infty(M)$ formed by normalized functions, with the map $F \mapsto \phi_F$ being viewed as the exponential map and the natural action of $\text{Ham}(M)$ on $C^\infty(M)$ being viewed as the adjoint action.

Similarly to the closed case, for an open symplectic manifold (U^{2n}, ω) one can define $\text{Ham}(U)$ as the group formed by Hamiltonian symplectomorphisms generated by (time-dependent) Hamiltonians supported in U and $\widetilde{\text{Ham}}(U)$ as its universal cover. The group $\widetilde{\text{Ham}}(U)$ admits the *Calabi homomorphism* $\widehat{\text{Cal}}_U :$

$\widetilde{Ham}(U) \rightarrow \mathbb{R}$ defined by $\widetilde{Cal}_U(\tilde{\phi}_F) := \int_{S^1} \int_M F_t \omega^n dt$, where $\text{supp } F \subset U$. If ω is exact, \widetilde{Cal}_U descends to a homomorphism $Cal_U : Ham(U) \rightarrow \mathbb{R}$. If U is an open subset of M , then there are natural inclusion homomorphisms $Ham(U) \rightarrow Ham(M)$, $\widetilde{Ham}(U) \rightarrow \widetilde{Ham}(M)$, whose images will be denoted by G_U and \tilde{G}_U ⁴.

Let U be an open subset of M . Each $\phi \in Ham(M)$ (respectively, $\tilde{\phi} \in \widetilde{Ham}(M)$) can be represented as a product of elements of the form $\psi\theta\psi^{-1}$ with θ lying in G_U (respectively, \tilde{G}_U). Moreover, assuming that \widetilde{Cal}_U descends to a homomorphism Cal_U on $Ham(U)$, one can make sure that each such θ satisfies $Cal_U(\theta) = 0$. This follows from Banyaga's fragmentation lemma [5]. Denote the minimal number of factors in such a product by $\|\phi\|_U$ (respectively, $\|\tilde{\phi}\|_U$), if there is no condition on $Cal_U(\theta)$, and $\|\phi\|_{U,0}$, if the condition $Cal_U(\theta) = 0$ is imposed. All the norms are defined as 0 on the identity elements.

Let T^k be a torus. A *Hamiltonian T^k -action* on M is a homomorphism $T^k \rightarrow Ham(M)$. (We will always assume that such an action is effective). In such a case the action of the i -th S^1 -factor of $T^k = S^1 \times \dots \times S^1$, $i = 1, \dots, k$, is a Hamiltonian flow generated by a Hamiltonian H_i . The Hamiltonians H_1, \dots, H_k commute with respect to the Poisson bracket. The map $\Phi = (H_1, \dots, H_k) : M \rightarrow \mathbb{R}^k$ is called the *moment map* of the Hamiltonian T^k -action. If all H_i are normalized, we say that Φ is the normalized moment map.

A submanifold L of (M^{2n}, ω) is called *Lagrangian*, if $\dim L = n$ and $\omega|_L \equiv 0$.

A (closed) symplectic manifold (M, ω) admits a preferred class of almost complex structures compatible in a certain sense with ω . All these almost complex structures have the same first Chern class c_1 , called the first Chern class of M . A closed symplectic manifold (M, ω) is called *monotone*, if $[\omega]$ and c_1 are positively proportional on spherical homology classes and *symplectically aspherical*, if $[\omega]$ vanishes on such classes.

Finally, we say that a subset $X \subset M$ is *displaceable from $Y \subset M$ by a group G* (where G is either $Ham(M)$, or $Symp_0(M)$, or $Symp(M)$), if there exists $\phi \in G$ such that $\phi(X) \cap \overline{Y} = \emptyset$. If X can be displaced from itself by G , we say that it is *displaceable by G* or just *displaceable*, if $G = Ham(M)$.

Consider $T^*S^1 = \mathbb{R} \times S^1$ with the coordinates (r, θ) and the symplectic form $dr \wedge d\theta$. We say that $X \subset M$ is *stably displaceable*, if $X \times \{r = 0\}$ is displaceable in $M \times T^*S^1$ equipped with the split symplectic form $\omega \oplus (dr \wedge d\theta)$. Any displaceable set is stably displaceable (but not necessarily vice versa).

3.2. Quantum homology and spectral numbers. A closed symplectic manifold M carries a rich algebraic structure called the *quantum homology* of M : additively it is just the singular homology of M with coefficients in a certain ring, while multiplicatively the quantum product is a deformation of the classical intersection product on homology which is defined using a count of certain pseudo-

⁴Note that the homomorphism $\widetilde{Ham}(U) \rightarrow \widetilde{Ham}(M)$ does not have to be injective and, accordingly, \tilde{G}_U does not have to be the preimage of G_U under the universal cover $\widetilde{Ham}(M) \rightarrow Ham(M)$ – see [40].

holomorphic spheres⁵ in M . In fact, there are several possible algebraic setups for this structure – we refer the reader to [48], as well as [22], [70], for the precise definitions and more details. In any case, the resulting algebraic object is a ring with unity given by the fundamental class $[M]$. Passing, if needed (depending on M and the algebraic setup of the construction), to an appropriate subring with unity one gets a finite-dimensional commutative algebra with unity over a certain field that we will denote by \mathcal{K} . Typically, \mathcal{K} is the field of semi-infinite (Laurent-type) power series with coefficients in a base field \mathcal{F} , where \mathcal{F} is one of the fields $\mathbb{Z}_p, \mathbb{Q}, \mathbb{R}, \mathbb{C}$. Abusing the terminology we will denote the latter finite-dimensional commutative algebra by $QH(M)$ and still call it the quantum homology of M .

Let us also mention the constructions of Usher [70] and Fukaya-Oh-Ohta-Ono [31] (the so-called *deformed quantum homology*) that, roughly speaking, use certain homology classes of M for an additional deformation of the quantum homology product and sometimes allow to obtain *different* finite-dimensional commutative algebras as above for a given M – abusing the terminology we will still call any of these different algebras the quantum homology of M and denote it by $QH(M)$ and emphasize the difference between them only when needed.

Given a non-zero $a \in QH(M)$ and a Hamiltonian $F : M \times S^1 \rightarrow \mathbb{R}$, one can define the *spectral number* $c(a, F)$ [61], [53] (see [51, 52, 71] for earlier versions of the construction and [68, 69] for additional important properties of the spectral numbers). It generalizes the following classical minimax quantity: given a singular non-zero (rational) homology class a of M and a continuous function F on M , consider the smallest value c of F so that a can be realized by a cycle lying in $\{F \leq c\}$ – for a smooth Morse function F this definition can be reformulated in terms of the Morse homology of F . The construction of the spectral number $c(a, F)$ is based on the same concept, where the singular homology is replaced by the quantum homology of M and the Morse homology of F is replaced by its *Floer homology*. The latter can be viewed as an infinite-dimensional version of the Morse homology for a certain functional, associated with F , on a covering of the space of free contractible loops in M , with the critical points of the functional being pre-images of contractible 1-periodic orbits of the Hamiltonian flow of F under the covering (see e.g. [48] for a detailed introduction to the subject).

If $F, G : M \times S^1 \rightarrow \mathbb{R}$ are normalized and $\tilde{\phi}_F = \tilde{\phi}_G$, then $c(a, F) = c(a, G)$. Thus, given $\tilde{\phi} \in \widetilde{Ham}(M)$, one can define $c(a, \tilde{\phi}) := c(a, F)$ for any normalized F generating $\tilde{\phi}$.

3.3. The main theorem. Assume $a \in QH(M)$ is an idempotent (for instance, $a = [M]$). **Here and further on, whenever we mention an idempotent, we assume that it is non-zero.** Define $\mu_a : \widetilde{Ham}(M) \rightarrow \mathbb{R}$ by

$$\mu_a(\tilde{\phi}) := -\text{vol}(M) \lim_{k \rightarrow +\infty} \frac{c(a, \tilde{\phi}^k)}{k},$$

⁵Pseudo-holomorphic spheres are (j, J) -holomorphic maps $(\mathbb{C}P^1, j) \rightarrow (M, J)$ for the standard complex structure j on $\mathbb{C}P^1$ and an almost complex structure J on M compatible with the symplectic form.

where $\text{vol}(M) := \int_M \omega^n$, and $\zeta_a : C^\infty(M) \rightarrow \mathbb{R}$ by

$$\zeta_a(F) := \lim_{k \rightarrow +\infty} \frac{c(a, kF)}{k}.$$

One can check [21] that the limits exist and

$$\zeta_a(F) = \frac{\int_M F \omega^n - \mu_a(\tilde{\phi}_F)}{\text{vol}(M)}.$$

The next theorem shows that under certain conditions on $QH(M)$ and a the function μ_a is a homogeneous quasi-morphism and accordingly ζ_a is a Lie quasi-state invariant under the adjoint action of $\widetilde{Ham}(M)$ on $C^\infty(M)$, since, up to a scaling factor and an addition of a linear map invariant under the adjoint action of $\widetilde{Ham}(M)$, it is the derivative of μ_a (see Section 2.3).

We will say that $QH(M)$ is *field-split*, if it can be represented, in the category of \mathcal{K} -algebras, as a direct sum of two subalgebras at least one of which is a field. Such a field will be called a *field factor* of $QH(M)$.

Theorem 3.1. *Assume $QH(M)$ is field-split and a is the unity in a field factor of $QH(M)$. Then μ_a satisfies the following properties:*

- A. **(Stability)** $\int_0^1 \min_M (F_t - G_t) dt \leq \mu_a(\tilde{\phi}_G) - \mu_a(\tilde{\phi}_F) \leq \int_0^1 \max_M (F_t - G_t) dt$.
- B. The function $\mu_a : \widetilde{Ham}(M) \rightarrow \mathbb{R}$ is a homogeneous quasi-morphism, that is,
 - B1. **(Homogeneity)** $\mu_a(\tilde{\phi}^k) = k\mu_a(\tilde{\phi})$ for any $\tilde{\phi} \in \widetilde{Ham}(M)$ and $k \in \mathbb{Z}$.
 - B2. **(Quasi-additivity)** There exists $C > 0$ such that $|\mu_a(\tilde{\phi}\tilde{\psi}) - \mu_a(\tilde{\phi}) - \mu_a(\tilde{\psi})| \leq C$ for any $\tilde{\phi}, \tilde{\psi} \in \widetilde{Ham}(M)$.
- C. **(Calabi property)** If $U \subset M$ is stably displaceable and $\text{supp } F \subset U$, then $\mu_a(\tilde{\phi}_F) = \int_0^1 \int_U F_t \omega^n dt$. In other words, the Calabi homomorphism \widetilde{Cal}_U descends from $\widetilde{Ham}(U)$ to $\tilde{G}_U \subset \widetilde{Ham}(M)$ ⁶ and $\mu|_{\tilde{G}_U} = \widetilde{Cal}_U$.

At the same time, ζ_a satisfies the following properties:

- a. **(Monotonicity)** $\min_M (F - G) \leq \zeta_a(F) - \zeta_a(G) \leq \max_M (F - G)$ for any $F, G \in C^\infty(M)$ and, in particular, if $F \leq G$, then $\zeta_a(F) \leq \zeta_a(G)$. Hence, ζ_a is 1-Lipschitz with respect to the uniform norm and extends to a functional on $C(M)$ that we will still denote by ζ_a .
- b. The functional ζ_a is a Lie quasi-state, that is
 - b1. **(Homogeneity)** $\zeta_a(\lambda F) = \lambda \zeta_a(F)$ for any $F \in C(M)$ and $\lambda \in \mathbb{R}$.
 - b2. **(Strong quasi-additivity)** If $F, G \in C^\infty(M)$ and $\{F, G\} = 0$, then $\zeta_a(F + G) = \zeta_a(F) + \zeta_a(G)$. In fact, ζ_a satisfies a stronger property: for any $F, G \in C^\infty(M)$ one has

$$|\zeta_a(F + G) - \zeta_a(F) - \zeta_a(G)| \leq \sqrt{2C||\{F, G\}||},$$

⁶This was tacitly assumed in [20].

where $C > 0$ is the constant from B2.

c. **(Vanishing property)** If $U \subset M$ is stably displaceable and $F \in C(M)$ with $\text{supp } F \subset U$, then $\zeta_a(F) = 0$.

d. **(Normalization)** $\zeta_a(1) = 1$.

e. **(Invariance)** $\zeta_a : C(M) \rightarrow \mathbb{R}$ is invariant under the action of $\text{Symp}_0(M)$ on $C(M)$.

The functionals $\mu_a : \widetilde{\text{Ham}}(M) \rightarrow \mathbb{R}$ and $\zeta_a : C(M) \rightarrow \mathbb{R}$ satisfying the properties listed in Theorem 3.1 are called, respectively, a *Calabi quasi-morphism* and a *symplectic quasi-state*. In particular, the restriction of a symplectic quasi-state to $C^\infty(M)$ is always a Lie quasi-state. Moreover, one can readily check that any symplectic quasi-state is also a topological quasi-state. The converse is true only if $\dim M = 2$ [21]. The quasi-measure associated to a symplectic quasi-state is $\text{Symp}_0(M)$ -invariant and vanishes on stably displaceable open sets.

For an arbitrary M and an arbitrary idempotent $a \in QH(M)$ (for instance, $a = [M]$) one gets a weaker set of properties of μ_a and ζ_a .

Theorem 3.2. *Assume $a \in QH(M)$ is an arbitrary idempotent. Then μ_a satisfies the properties (A) and (C) from Theorem 3.1 and a weaker version of the properties (B1) and (B2):*

B1'. **(Partial homogeneity)** $\mu_a(\tilde{\phi}^k) = k\mu_a(\tilde{\phi})$ for any $\tilde{\phi} \in \widetilde{\text{Ham}}(M)$ and $k \in \mathbb{Z}_{\geq 0}$.

B2'. **(Partial quasi-additivity)** Given a displaceable open set $U \subset M$, there exists $C = C(\mu_a, U) > 0$, so that $|\mu_a(\tilde{\phi}\tilde{\psi}) - \mu_a(\tilde{\phi}) - \mu_a(\tilde{\psi})| \leq C \min\{\|\tilde{\phi}\|_U, \|\tilde{\psi}\|_U\}$ for any $\tilde{\phi}, \tilde{\psi} \in \widetilde{\text{Ham}}(M)$.

At the same time, ζ_a satisfies the properties (a),(c),(d),(e) from Theorem 3.1 and a weaker version of the properties (b1) and (b2):

b1'. **(Partial homogeneity)** $\zeta_a(\lambda F) = \lambda\zeta_a(F)$ for any $F \in C(M)$ and $\lambda \in \mathbb{R}_{\geq 0}$.

b2'. **(Partial strong quasi-additivity)** If $F, G \in C^\infty(M)$ and $\{F, G\} = 0$ and either $\text{supp } G$ is displaceable or G is constant, then $\zeta_a(F + G) = \zeta_a(F) + \zeta_a(G)$. In fact, ζ_a satisfies a stronger property: for any $F, G \in C^\infty(M)$ one has

$$|\zeta_a(F + G) - \zeta_a(F) - \zeta_a(G)| \leq \sqrt{2C\|\{F, G\}\|},$$

where $C \in (0, +\infty]$ is a constant depending on ζ_a and on the supports of F and G . In particular, the constant C is finite if at least one of the supports is displaceable.

The functionals μ_a and ζ_a satisfying the properties listed in Theorem 3.2 are called, respectively, a *partial Calabi quasi-morphism* and a *partial symplectic quasi-state*. Clearly, a genuine Calabi quasi-morphism or a genuine symplectic quasi-state is also a partial one.

Theorems 3.1 and 3.2 were proved under various additional restrictions on M in [20], [21]. In [54, 45, 68, 69] (see also [22, 23]) the restrictions were removed

and the theorems were proved for new classes of closed symplectic manifolds. The stronger part of the property (b2) in Theorems 3.1 was proved in [28]. The stronger part of the property (b2') in Theorems 3.2 was proved in [57]. The Calabi and vanishing properties (C) and (c) in Theorems 3.1, 3.2 were originally proved for a displaceable U – it was later observed by Borman [10] that the displaceability assumption on U can be weakened to stable displaceability.

Examples of closed manifolds with a field-split $QH(M)$ (for an appropriate algebraic setup of $QH(M)$) include complex projective spaces (or, more generally, complex Grassmanians and symplectic toric manifolds), as well as blow-ups of symplectic manifolds – all with appropriate (and, in a certain sense, generic [70]) symplectic structures [20, 22, 31, 32, 44, 45, 54, 55, 70]. The direct product of symplectic manifolds with field-split quantum homology algebras also has this property – possibly under some additional assumptions on the manifolds, depending on the algebraic setup of the quantum homology [22]. As an example of M whose quantum homology $QH(M)$ is *not* field-split one can take any symplectically aspherical manifold – in such a case there are no pseudo-holomorphic spheres in M and hence there is no difference between quantum and singular homology. Let us emphasize that, in general, the question whether $QH(M)$ is field-split may depend not only on M but also on the algebraic setup of the quantum homology (and on a choice of the deformation in case of the deformed quantum homology).

Example 3.3 ([20]). Assume $M = S^2$ and $a = [S^2]$. Then $\zeta_a : C(S^2) \rightarrow \mathbb{R}$ is a symplectic quasi-state and its restriction to the set of smooth Morse functions on S^2 (which is dense in $C(S^2)$ in the uniform norm) can be described in combinatorial terms.

Namely, assume that the symplectic (that is, area) form ω on S^2 is normalized so that the area of S^2 is 1 and let F be a smooth Morse function on S^2 . Consider the space Δ of connected components of the level sets of F as a quotient space of S^2 . As a topological space Δ is homeomorphic to a tree. The function F descends to Δ and the push-forward of the measure defined by ω on S^2 yields a non-atomic Borel probability measure on Δ . There exists a unique point $x \in \Delta$ such that each connected component of $\Delta \setminus x$ has measure $\leq 1/2$ (such a point x is called *the median* of the measured tree Δ). Then $\zeta_a(F) = F(x)$.

Let us note that the symplectic quasi-state $\zeta_a : C(S^2) \rightarrow \mathbb{R}$ in Example 3.3 is *dispersion-free*, that is, satisfies $\zeta_a(F^2) = (\zeta_a(F))^2$. Equivalently, the corresponding quasi-measure takes only values 0 and 1. The following open question is of utmost importance for the study of symplectic quasi-states:

Question 3.4. Is it true that the symplectic quasi-states constructed in Theorems 3.1 and 3.2 are always dispersion-free?

Remark 3.5. Sometimes the (partial) Calabi quasi-morphism $\mu_a : \widetilde{Ham}(M) \rightarrow \mathbb{R}$ descends to $Ham(M)$ (that is, vanishes on $\pi_1 Ham(M)$). Abusing the notation we will denote the resulting (partial) Calabi quasi-morphism on $Ham(M)$ also by μ_a . The list of manifolds for which μ_a is known to descend to $Ham(M)$ for *all* a includes symplectically aspherical manifolds [61], complex projective spaces [20]

and their monotone products [20, 13], a monotone blow-up of $\mathbb{C}P^2$ at three points and the complex Grassmannian $Gr(2, 4)$ [13]. The list of manifolds for which it is known that μ_a does *not* descend to $Ham(M)$ at least for *some* a includes various symplectic toric manifolds and, in particular, the monotone blow-ups of $\mathbb{C}P^2$ at one or two points [23, 54].

Let us note that if (M, ω) is monotone, the restriction of μ_a to $\pi_1 Ham(M)$ does not depend on the choice of a (for a fixed algebraic setup of $QH(M)$) [23]. This is not necessarily true if M is not monotone [55].

Remark 3.6. For a closed connected M the group $Ham(M)$ is simple and the group $\widetilde{Ham}(M)$ is perfect [5]. Hence, these groups do not admit non-trivial homomorphisms to \mathbb{R} and therefore partial Calabi quasi-morphisms on $\widetilde{Ham}(M)$ and $Ham(M)$ are never homomorphisms (they are non-trivial because of the Calabi property). Also, partial symplectic quasi-states are never linear (use a partition of unity with displaceable supports to check it). Moreover, in certain cases one can verify that a partial symplectic quasi-state ζ_a is not a genuine quasi-state (and, accordingly, μ_a is not a genuine quasi-morphism) – see Section 4.1.

Remark 3.7. Denote by \mathcal{E} the collection of all open displaceable $U \subset (M, \omega)$ such that $\omega|_U$ is exact. For any $U \in \mathcal{E}$ the Calabi homomorphism $Cal_U : G_U \rightarrow \mathbb{R}$ is well-defined and, by Banyaga's theorem [5], the group $\text{Ker } Cal_U$ is simple, meaning that, up to a scalar factor, Cal_U is the unique non-trivial \mathbb{R} -valued homomorphism on G_U continuous on 1-parametric subgroups. If $U, V \in \mathcal{E}$ and $U \subset V$, then $G_U \subset G_V$ and $Cal_U = Cal_V$ on G_U . Thus, if a partial Calabi quasi-morphism μ_a descends to $Ham(M)$, we get the following picture: there is a family \mathcal{E} of subgroups of $Ham(M)$, with each subgroup $G_U \in \mathcal{E}$ carrying the unique \mathbb{R} -valued homomorphism Cal_U (continuous on 1-parametric subgroups), and while it is impossible to patch up all these homomorphisms into an \mathbb{R} -valued homomorphism on $Ham(M)$, it is possible to patch them up into a partial Calabi quasi-morphism μ_a (which may be non-unique).

Now let us discuss the existence and uniqueness of genuine Calabi quasi-morphisms and symplectic quasi-states on a given symplectic manifold and, in particular, the dependence of μ_a and ζ_a on a and the algebraic setup of $QH(M)$.

The set of idempotents in $QH(M)$ carries a partial order: namely, given idempotents $a, b \in QH(M)$, we write $a \succeq b$ if $ab = b$. Clearly, $[M] \geq b$ for any idempotent $b \in QH(M)$. If $a \succeq b$, then $a - b$ is also an idempotent and $a \succeq a - b$. Conversely, if $b, b' \in QH(M)$ are two idempotents such that $bb' = 0$, then $b + b'$ is also an idempotent and $b + b' \succeq b, b'$.

The following theorem follows from basic properties of spectral numbers (cf. [23], Theorem 1.5).

Theorem 3.8. *Assume $a, b \in QH(M)$ are idempotents, so that $a \succeq b$. Then*

- a. $\mu_a \leq \mu_b$, $\zeta_a \geq \zeta_b$.*
- b. If μ_a is a genuine (i.e. not only partial) Calabi quasi-morphism, then $\mu_a = \mu_b$, $\zeta_a = \zeta_b$ and thus μ_b is also a genuine Calabi quasi-morphism and ζ_b is a genuine symplectic quasi-state.*

At the same time it is possible that $a \succeq b$, μ_a is a partial Calabi quasi-morphism while μ_b is a genuine Calabi quasi-morphism – see Examples 4.13, 4.14.

In fact, different idempotents may define linearly independent Calabi quasi-morphisms and symplectic quasi-states. For instance, if M is a blow-up of $\mathbb{C}P^2$ at one point with an appropriate non-monotone symplectic structure, one can find Calabi quasi-morphisms μ_a and μ_b on $\widetilde{Ham}(M)$ (for some idempotents $a, b \in QH(M)$) that have linearly independent restrictions to $\pi_1 Ham(M)$ [55], and if M is $S^2 \times S^2$ with a monotone symplectic structure, one can find linearly independent⁷ symplectic quasi-states ζ_a, ζ_b on $C(M)$ – see Example 4.14.

Moreover, a change of the algebraic setup of $QH(M)$ may yield new Calabi quasi-morphisms and symplectic quasi-states on the same M . For instance, if $M = \mathbb{C}P^n$, one can choose algebraic setups of $QH(\mathbb{C}P^n)$, both for $\mathcal{F} = \mathbb{Z}_2$ and $\mathcal{F} = \mathbb{C}$, so that in both cases $QH(\mathbb{C}P^n)$ is a field and thus the *only* idempotent in $QH(\mathbb{C}P^n)$ is the unity $a = [\mathbb{C}P^n]$. By Theorem 3.1, in both cases $a = [\mathbb{C}P^n]$ defines a Calabi quasi-morphism μ_a on $\widetilde{Ham}(\mathbb{C}P^n)$ that descends to $Ham(\mathbb{C}P^n)$ (for $\mathcal{F} = \mathbb{C}$ this is proved in [20], the same proof works also for $\mathcal{F} = \mathbb{Z}_2$). However, it follows from [73] for $n = 2$ and from [50] for $n = 3$ that the symplectic quasi-states defined by $[\mathbb{C}P^n]$ in both cases are *different* (see Example 4.12).

Remark 3.9. Let us note that historically the first Calabi quasi-morphism on $\widetilde{Ham}(\mathbb{C}P^n)$ was implicitly constructed by Givental in [34], [35] in a completely different way; the fact that it is indeed a Calabi quasi-morphism was proved by Ben Simon [7] (the stability property of the quasi-morphism is proved in [12]). Givental's Calabi quasi-morphism descends from $\widetilde{Ham}(\mathbb{C}P^n)$ to $Ham(\mathbb{C}P^n)$ [64]. It would be interesting to find out whether this quasi-morphism on $Ham(\mathbb{C}P^n)$ can be expressed as μ_a for some $a \in QH(\mathbb{C}P^n)$ (for some algebraic setup of $QH(\mathbb{C}P^n)$).

Question 3.10. Is the quasi-morphism μ_a for $a = [\mathbb{C}P^1]$ the only Calabi quasi-morphism on $Ham(\mathbb{C}P^1)$?

Let us note that in the case of $\mathbb{C}P^1$ the symplectic quasi-state ζ_a for $a = [\mathbb{C}P^1]$, described in Example 3.3, is known to be the unique symplectic quasi-state on $C(S^2)$ [20].

In some cases the non-uniqueness can be prove by using the different algebras $QH(M)$ appearing in the deformed quantum homology construction – see [31] for examples of symplectic manifolds with infinitely many linearly independent Calabi quasi-morphisms and symplectic quasi-states constructed in this way.

Let us also mention a construction due to Borman [10],[11] (also see [12]) that allows in certain cases to use a Calabi quasi-morphism on $\widetilde{Ham}(N)$ in order to build a Calabi quasi-morphism on $\widetilde{Ham}(M)$ if M is obtained from N by a symplectic reduction or if M is a symplectic submanifold of N . Using different presentations of a symplectic manifold as a symplectic reduction one can construct examples of

⁷Note that Calabi quasi-morphisms and symplectic quasi-states on a given manifold form convex sets respectively in the vector spaces of all homogeneous quasi-morphisms and all Lie quasi-states and their linear dependence is considered in these vector spaces.

M with infinitely many linearly independent Calabi quasi-morphisms on $\widetilde{Ham}(M)$ (the corresponding symplectic quasi-states are also linearly independent).

Let us also note that apart from the Calabi quasi-morphisms from Theorem 3.1, the Givental quasi-morphism mentioned in Remark 3.9 and the quasi-morphisms produced from them by Borman's reduction method, there are no known examples of homogeneous quasi-morphisms on $\widetilde{Ham}(M)$ satisfying the stability property (A) from Theorem 3.1, though otherwise there are many homogeneous quasi-morphisms on $\widetilde{Ham}(M)$ (that sometimes descend to $Ham(M)$) – see [19, 33, 59, 60, 65]. Let us also note that for symplectic manifolds of dimension greater than 2 the constructions above are the only currently known constructions of partial symplectic quasi-states on closed manifolds. (As it was mentioned above, in dimension 2 any topological quasi-state is also symplectic. There is a number of ways to construct topological quasi-states in any dimension – see e.g. [2, 41]). The basic open case for the existence of a (stable) Calabi quasi-morphism on $\widetilde{Ham}(M)$ is $M = T^2$ and for symplectic quasi-states it is $M = T^4$. It is also unknown whether any symplectic quasi-state has to come from a Calabi quasi-morphism and whether different Calabi quasi-morphisms may define the same symplectic quasi-state.

Remark 3.11. There is a straightforward extension of the notion of a (partial) Calabi quasi-morphism to the case of an open symplectic manifold M and there are several constructions of such quasi-morphisms.

First, one can consider a conformally symplectic embedding⁸ of M in a closed symplectic manifold N carrying a Calabi quasi-morphism defined on $Ham(N)$ and use the homomorphism $Ham(M) \rightarrow Ham(N)$ induced by the embedding to pullback the quasi-morphism from $Ham(N)$ to $Ham(M)$. (Of course, one then has to prove that the resulting quasi-morphism on $Ham(M)$ is non-trivial). In this way one can, for instance, use conformally symplectic embeddings of a standard round ball B^{2n} in $\mathbb{C}P^n$ in order to construct a continuum of linearly independent Calabi quasi-morphisms on $Ham(B^{2n})$ [9].

There are also *intrinsic* constructions of (partial) Calabi quasi-morphisms for certain open symplectic manifolds following the lines of the construction presented above – see [42, 49] for more details as well as for extensions of the notion of a (partial) symplectic quasi-state to the open case. These constructions allow to extend many of the results mentioned in this survey to the open case.

4. Applications

4.1. Quasi-states and rigidity of symplectic intersections. A key phenomenon in symplectic topology is rigidity of intersections of subsets of symplectic manifolds: namely, sometimes a subset X of a symplectic manifold M cannot be displaced from a subset Y by $Ham(M)$ (or $Symp_0(M)$, or $Symp(M)$), even though X can be displaced from Y by a smooth isotopy. A central role in

⁸A map $f : (M, \omega) \rightarrow (N, \Omega)$ between symplectic manifolds is called *conformally symplectic* if $f^*\Omega = c\omega$ for some non-zero constant c .

the applications of partial symplectic quasi-states is played by their connection to this phenomenon. Namely, to each partial symplectic quasi-state, and, in particular, to each idempotent $a \in QH(M)$, one can associate a certain hierarchy of non-displaceable sets in M . The interplay between the hierarchies associated to different a is an interesting geometric phenomenon in itself.

The key definitions describing the hierarchy are as follows [23]. Let $\zeta : C(M) \rightarrow \mathbb{R}$ be a partial symplectic quasi-state. We say that a closed subset $X \subset M$ is *heavy with respect to ζ* , if $\zeta(F) \geq \inf_X F$ for all $F \in C(M)$, and *superheavy with respect to ζ* , if $\zeta(F) \leq \sup_X F$ for all $F \in C(M)$. Equivalently, X is superheavy with respect to ζ , if $\zeta(F) = F(X)$ for any $F \in C(M)$ which is constant on X . If $\zeta = \zeta_a$ for an idempotent $a \in QH(M)$ (and a prefixed algebraic setup of $QH(M)$), we use the terms *a-heavy* and *a-superheavy* for the heavy and superheavy sets with respect to ζ_a . Clearly, a closed set containing a heavy/superheavy subset is itself heavy/superheavy. The basic properties of heavy and superheavy sets are summarized in the following theorems.

Theorem 4.1 ([23]). *Heavy and superheavy sets with respect to a fixed partial symplectic quasi-state ζ satisfy the following properties:*

- a. Every superheavy set is heavy, but, in general, not vice versa.*
- b. The classes of heavy and superheavy sets are $\text{Symp}_0(M)$ -invariant.*
- c. Every superheavy set has to intersect every heavy set. Therefore, in view of (b), any superheavy set cannot be displaced from any heavy set by $\text{Symp}_0(M)$. In particular, any superheavy set is non-displaceable by $\text{Symp}_0(M)$. On the other hand, two heavy sets may be disjoint.*
- d. Every heavy subset is stably non-displaceable. However, it may be displaceable by $\text{Symp}_0(M)$.*
- e. If ζ is a genuine (that is, not partial) symplectic quasi-state, then the classes of heavy and superheavy sets are identical: they coincide with the class of closed sets of full quasi-measure (that is, of quasi-measure 1) for the quasi-measure on M associated with ζ .*

Theorem 4.2 ([23]). *Assume that X_i is an a_i -heavy (resp. a_i -superheavy) subset of a closed connected symplectic manifold M_i for some idempotent $a_i \in QH(M_i)$, $i = 1, 2$. Then the product $X_1 \times X_2 \subset M_1 \times M_2$ is $a_1 \otimes a_2$ -heavy (resp. $a_1 \otimes a_2$ -superheavy)⁹.*

Changing an idempotent a or changing the algebraic setup of $QH(M)$ may completely change the heaviness/superheaviness property of a set: there are examples of disjoint sets that are superheavy with respect to different idempotents in $QH(M)$ (see Example 4.14) and there is an example of a set that is $[M]$ -superheavy, if $QH(M)$ is set up over $\mathcal{F} = \mathbb{Z}_2$, and is disjoint from an $[M]$ -superheavy set, if $QH(M)$ is set up over $\mathcal{F} = \mathbb{C}$ (see Example 4.12).

⁹There is an analogue of the Künneth formula for quantum homology – in particular, to each pair of idempotents $a_1 \in QH(M_1)$, $a_2 \in QH(M_2)$, one can associate an idempotent $a_1 \otimes a_2 \in QH(M_1 \times M_2)$. Let us note that even if ζ_{a_1} and ζ_{a_2} are genuine symplectic quasi-states, $\zeta_{a_1 \otimes a_2}$ may be only a partial one – see Example 4.14.

The partial order on the set of idempotents mentioned in Section 3 yields the following relation between the corresponding collections of heavy and superheavy sets which follows immediately from Theorem 3.8 and the examples below.

Theorem 4.3. *Assume $a, b \in QH(M)$ are idempotents and $a \succeq b$. Then*

- a. Every a -superheavy set is also b -superheavy (but not necessarily vice versa).*
- b. Every b -heavy set is also a -heavy (but not necessarily vice versa).*

Remark 4.4. There is a natural action of $Symp(M)$ on $QH(M)$. The subgroup $Symp_0(M) \subset Symp(M)$ acts on $QH(M)$ trivially and this explains the $Symp_0(M)$ -invariance of the partial symplectic quasi-states ζ_a and, accordingly, of the classes of a -heavy and a -superheavy sets. If an idempotent $a \in QH(M)$ is invariant under the action of the full group $Symp(M)$ (e.g. if $a = [M]$), then $Symp_0(M)$ can be replaced by $Symp(M)$ everywhere in Theorems 3.1, 3.2, 4.1. In particular, *any* $[M]$ -superheavy set cannot be displaced from *any* a -heavy set (for *any* idempotent $a \in QH(M)$) by $Symp(M)$.

Remark 4.5. The reason why every superheavy set X (with respect to a partial quasi-state ζ) must intersect every heavy set Y is very simple: If $X \cap Y = \emptyset$, pick a function F so that $F|_X \equiv 0$ and $F|_Y \equiv 1$. Then, by the definition of heavy and superheavy sets, $\zeta(F) = 0$ and $\zeta(F) \geq 1$, which is impossible.

Remark 4.6. For certain Lagrangian submanifolds X and Y of M one can prove the non-displaceability of X from Y by means of the Lagrangian Floer homology of the pair (X, Y) (see e.g. [30]). The advantage of this method is that, unlike Theorem 4.1, it gives a non-trivial lower bound on the number of transverse intersection points of $\phi(X)$, $\phi \in Ham(M)$, and Y . On the other hand, unlike the Lagrangian Floer theory, symplectic quasi-states allow to prove non-displaceability results for singular sets (see the examples below).

The most basic examples of heavy and superheavy sets are an equator in S^2 (that is, an embedded circle dividing S^2 into two parts of equal area) which is $[S^2]$ -superheavy and a meridian in T^2 which is $[T^2]$ -heavy but not $[T^2]$ -superheavy, since it is displaceable by $Symp_0(T^2)$ – in particular, it implies that the partial symplectic quasi-state on $C(T^2)$ defined by $[T^2]$ is not a genuine symplectic quasi-state [23] (cf. Remark 3.6). The union of a meridian and a parallel in T^2 is $[T^2]$ -superheavy [38]. More complicated examples come from the following constructions.

Let $\mathbb{A} \subset C^\infty(M)$ be a finite-dimensional Poisson-commutative vector subspace (meaning that $\{F, G\} = 0$ for any $F, G \in \mathbb{A}$). The map $\Phi : M \rightarrow \mathbb{A}^*$ defined by $\langle \Phi(x), F \rangle = F(x)$ is called *the moment map* of \mathbb{A} . As an example of such a moment map one can consider the moment map of a Hamiltonian torus action on M , or a map $M \rightarrow \mathbb{R}^N$ whose components have disjoint supports.

A non-empty fiber $\Phi^{-1}(p)$ is called a *stem* of \mathbb{A} (see [21]), if all non-empty fibers $\Phi^{-1}(q)$ with $q \neq p$ are displaceable, and a *stable stem*, if they are stably displaceable. If a subset of M is a (stable) stem of *some* finite-dimensional Poisson-commutative subspace of $C^\infty(M)$, it will be called just a (stable) *stem*. Any stem is a stable stem but possibly not vice versa ¹⁰.

¹⁰There are no known examples of a stable stem that is not a stem, i.e. not a stem of *any* \mathbb{A} .

Theorem 4.7 ([21, 23]). *A stable stem is superheavy with respect to any partial symplectic quasi-state ζ on $C(M)$.*

Using the partial symplectic quasi-state $\zeta_{[M]}$ we get

Corollary 4.8 ([21]). *For any finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ its moment map Φ has at least one non-displaceable fiber.*

The following question is closely related to Question 3.4.

Question 4.9. Is it true that for any finite-dimensional Poisson-commutative subspace of $C^\infty(M)$ its moment map Φ has at least one *heavy* fiber (at least with respect to *some* symplectic quasi-state ζ on $C(M)$)?

Remark 4.10. If ζ a genuine symplectic quasi-state on $C(M)$, then Theorem 4.7 can be proved using the quasi-measure τ associated to ζ [21]. Namely, the push-forward of τ to \mathbb{A}^* by the moment map Φ of a Poisson commutative subspace \mathbb{A} is a Borel probability measure ν on \mathbb{A}^* [21]. As we already noted above, the vanishing property of ζ implies that τ vanishes on stably displaceable open subsets of M . Therefore if a fiber $\Phi^{-1}(p)$ of Φ is a stable stem, the support of ν has to be concentrated at p , meaning that $\tau(\Phi^{-1}(p)) = 1$ or, in other words (see Theorem 4.1), $\Phi^{-1}(p)$ is superheavy with respect to ζ .

Here are a few examples of (stable) stems [21, 23]. The Lagrangian Clifford torus in $\mathbb{C}P^n$, defined as $L = \{[z_0 : \dots : z_n] \in \mathbb{C}P^n \mid |z_0| = \dots = |z_n|\}$, is a stem, hence $[\mathbb{C}P^n]$ -superheavy (this generalizes the example of an equator in S^2 mentioned above). The codimension-1 skeleton of a triangulation of a closed symplectic manifold M^{2n} all of whose $2n$ -dimensional simplices are displaceable is a stem. A fiber $\Phi^{-1}(0)$ of the normalized moment map Φ of a compressible¹¹ Hamiltonian torus action on M is a stable stem¹². A direct product of (stable) stems is a (stable) stem and that the image of a (stable) stem under *any* symplectomorphism is again a (stable) stem.

In case when the Hamiltonian torus action is not compressible, much less is known about (stable) displaceability of fibers of the moment map of the action (aside from the case of symplectic toric manifolds where many results have been obtained in recent years by different authors). If (M, ω) is monotone, one can explicitly find a so-called *special fiber* of the moment map Φ of the action which is a -superheavy for *any* idempotent $a \in QH(M)$ [23]. For a Hamiltonian T^n -action torus on a monotone (M^{2n}, ω) (that is, for a monotone symplectic toric manifold) the special fiber (which in this case is a Lagrangian torus) can be described in simple combinatorial terms involving the moment polytope (that is, the image of the moment map which is a convex polytope) – see [23]. It is not known whether in the latter case the special fiber is always a stem – see [46] for a detailed investigation of this question. Interestingly enough, the question whether the special fiber of the *normalized* moment map for a monotone symplectic toric manifold M coincides

¹¹An effective Hamiltonian T^k -action on (M, ω) is called *compressible* if the image of the homomorphism $\pi_1(T^k) \rightarrow \pi_1(\text{Ham}(M))$, induced by the action, is a finite group.

¹²Stable stems appearing in this way potentially may not be genuine stems.

with the fiber over zero is related to the existence of a Kähler-Einstein metric on M – see [23, 64].

Finally, heaviness/superheaviness of Lagrangian submanifolds can be proved using various versions of Lagrangian Floer homology. Namely, to certain Lagrangian submanifolds L of M one can associate the *quantum homology* (or the *Lagrangian Floer homology*) $QH(L)$ that comes with an *open-closed map* $i_L : QH(L) \rightarrow QH(M)$ [3, 8, 30, 31]. If $i_L(x)$ is non-zero for certain $x \in QH(L)$, then L is $[M]$ -heavy¹³, and if $i_L(x)$ divides an idempotent $a \in QH(M)$, then L is a -superheavy – this is shown in [23] in the monotone case, cf. [8, 30, 31].

Here are a few examples where this method can be applied. Let us emphasize that the applications to specific L do depend on a proper choice of the algebraic setup for the quantum homology in each case – see e.g. Example 4.12; we will ignore this issue in the other examples below and refer the reader to [8, 23, 31] for details.

Example 4.11 ([23]). Assume that $L \subset M$ is a Lagrangian submanifold and $\pi_2(M, L) = 0$. Then L is $[M]$ -heavy. Note that in this case heaviness may not be improved to superheaviness: the meridian in T^2 is $[T^2]$ -heavy but not $[T^2]$ -superheavy.

Example 4.12. The real projective space $\mathbb{R}P^n$, which is a Lagrangian submanifold of $\mathbb{C}P^n$, is $[\mathbb{C}P^n]$ -superheavy, *as long as* $QH(\mathbb{C}P^n)$ *is set up over* $\mathcal{F} = \mathbb{Z}_2$ [8, 23]. In particular, this implies that $\mathbb{R}P^n$ is not displaceable by $Symp(\mathbb{C}P^n)$ from the Clifford torus (see [4, 67] for other proofs of this fact).

On the other hand, $\mathbb{R}P^n$ may not be $[\mathbb{C}P^n]$ -superheavy, if $QH(\mathbb{C}P^n)$ is set up over $\mathcal{F} = \mathbb{C}$ – for $n = 2$ this follows from [73] and for $n = 3$ from [50]. This implies that the symplectic quasi-states defined by $[\mathbb{C}P^n]$ for the setups of $QH(M)$ over $\mathcal{F} = \mathbb{Z}_2$ and $\mathcal{F} = \mathbb{C}$ are different for $n = 2, 3$.

Example 4.13 ([23]). Consider the torus T^{2n} equipped with the standard symplectic structure $\omega = dp \wedge dq$. Let $M^{2n} = T^{2n} \# \overline{\mathbb{C}P^n}$ be a symplectic blow-up of T^{2n} at one point (the blow-up is performed in a small ball B around the point). Assume that the Lagrangian torus $L \subset T^{2n}$ given by $q = 0$ does not intersect B .

Then the proper transform of L is a Lagrangian submanifold of M which is $[M]$ -heavy but *not* a -heavy for some other idempotent $a \in QH(M)$ (that, roughly speaking, depends on the exceptional divisor of the blow-up). In this case the functional $\zeta_{[M]}$ is a partial (but not genuine) symplectic quasi-state while ζ_a is a genuine symplectic quasi-state.

Example 4.14 ([23, 18]). Let S^2 be the standard unit sphere in \mathbb{R}^3 with the standard area form σ . Let $M := S^2 \times S^2$ be equipped with the symplectic form $\sigma \oplus \sigma$. Denote by x_1, y_1, z_1 and x_2, y_2, z_2 the Euclidean coordinates on the two S^2 -factors.

Consider the following three Lagrangian submanifolds of M : the anti-diagonal $\Delta := \{(u, v) \in S^2 \times S^2 : u = -v\}$, the Clifford torus $L = \{z_1 = z_2 = 0\}$ and the

¹³The a -heaviness for an arbitrary idempotent $a \in QH(M)$ can also be proved by the same method under a stronger non-vanishing assumption.

torus $K = \{z_1 + z_2 = 0, x_1x_2 + y_1y_2 + z_1z_2 = -1/2\}$. Clearly, L intersects both K and Δ , while $K \cap \Delta = \emptyset$.

For a certain algebraic setup of $QH(M)$ (with $\mathcal{F} = \mathbb{C}$) the algebra $QH(M)$ is a direct sum of two fields whose unities will be denoted by a_- and a_+ (in particular, $a_- + a_+ = [M]$). The idempotents a_- and a_+ define symplectic quasi-states ζ_{a_-} , ζ_{a_+} . At the same time $\zeta_{[M]}$ is only a partial, not genuine, symplectic quasi-state. The submanifolds Δ and L are a_- -superheavy, while K is not. At the same time K and L are a_+ -superheavy, while Δ is not. All three sets Δ , K , L are $[M]$ -heavy but L is the only one of them that is $[M]$ -superheavy.

In particular, the Lagrangian tori L and K cannot be mapped into each other by any symplectomorphism of M . See [50] and the references therein for more results on the Lagrangian torus K .

For more examples of heavy and superheavy Lagrangian submanifolds obtained by means of an open-closed map see [23, 31].

4.2. Quasi-states and connecting trajectories of Hamiltonian flows. Here is an application of symplectic quasi-states to Hamiltonian dynamics. As above, we assume that M is a closed connected symplectic manifold.

Theorem 4.15 ([14]). *Let $X_0, X_1, Y_0, Y_1 \subset M$ be a quadruple of closed sets so that $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$ and the sets $X_0 \cup Y_0, Y_0 \cup X_1, X_1 \cup Y_1, Y_1 \cup X_0$ are all a -superheavy for an idempotent $a \in QH(M)$. Let $G \in C^\infty(M \times S^1)$ be a 1-periodic Hamiltonian with $G_t|_{Y_0} \leq 0$, $G_t|_{Y_1} \geq 1$ for all $t \in S^1$.*

Then there exists a point $x \in M$ and time moments $t_0, t_1 \in \mathbb{R}$ so that $\phi_G^{t_0}(x) \in X_0$ and $\phi_G^{t_1}(x) \in X_1$. Furthermore, $|t_0 - t_1|$ can be bounded from above by a constant depending only on a , if G is time-independent, and both on a and the oscillation $\max_{M \times S^1} G - \min_{M \times S^1} G$ of G , if G is time-dependent.

Remark 4.16. The proof of Theorem 4.15 uses the following important notion [14]: Given a quadruple X_0, Y_0, X_1, Y_1 of compact subsets of a (possibly open) symplectic manifold such that $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$, define $pb_4(X_0, Y_0, X_1, Y_1)$ (where pb stands for the ‘‘Poisson brackets’’ and 4 for the number of sets) by $pb_4(X_0, Y_0, X_1, Y_1) = \inf_{F, G} \|\{F, G\}\|$, where the infimum is taken over all compactly supported smooth F, G satisfying $F|_{X_0} \leq 0$, $F|_{X_1} \geq 1$, $G|_{Y_0} \leq 0$, $G|_{Y_1} \geq 1$.

Given a quadruple X_0, Y_0, X_1, Y_1 as in Theorem 4.15, one can use the strong form of the property (b2) in Theorem 3.1 to prove the positivity of pb_4 for certain stabilizations [14] of the four sets. The existence of a connecting trajectory of the Hamiltonian flow is then deduced from the positivity of pb_4 using an averaging argument [14].

Example 4.17 ([14]). Consider an open tubular neighborhood U of the zero-section T^n in T^*T^n . Pick $q_0, q_1 \in T^n$ and consider the open cotangent disks $D_i = T_{q_i}^*T^n \cap U$, $i = 0, 1$. Let $M = S^2 \times \dots \times S^2$ be the product of n copies of S^2 equipped with the split symplectic structure $\omega = \sigma \oplus \dots \oplus \sigma$. Let $Y_1 \subset M$ be the product of equators in the S^2 factors. It is a Lagrangian torus. If $\int_{S^2} \sigma$ is sufficiently large, then, by the Weinstein neighborhood theorem (see [47]), U

can be symplectically identified with a tubular neighborhood of Y_1 in M . Using the identification we consider $X_i := \overline{D}_i$, $i = 0, 1$, Y_1 and U as subsets of M . Set $Y_0 := M \setminus U$.

If $\int_{S^2} \sigma$ is sufficiently large, the quadruple $X_0, Y_0, X_1, Y_1 \subset M$ satisfies the assumptions of Theorem 4.15 [14]. In particular, let $G : T^*T^n \times S^1 \rightarrow \mathbb{R}$ be a Hamiltonian supported in $U \subset T^*T^n$ which is ≥ 1 on $T^n \times S^1$. Theorem 4.15 implies the existence of a trajectory of the Hamiltonian flow of G passing through D_0 and D_1 .

Switching the pair X_0, X_1 with the pair Y_0, Y_1 and applying Theorem 4.15 to the switched pairs one can show in a similar way that if $F : T^*T^n \times S^1 \rightarrow \mathbb{R}$ is a compactly supported Hamiltonian such that $F|_{D_0 \times S^1} \leq 0$, $F|_{D_1 \times S^1} \geq 1$, then there exists a trajectory of the Hamiltonian flow of F connecting the zero-section of T^*T^n with ∂U . It would be interesting to find out whether this fact can be related to the well-known Arnold diffusion phenomenon in Hamiltonian dynamics that concerns trajectories of a similar kind.

4.3. Quasi-states and C^0 -rigidity of Poisson brackets. In this section we still assume that (M, ω) is a closed connected symplectic manifold.

The Poisson brackets of two smooth functions on (M, ω) depend on their first derivatives. Nevertheless, as it was first discovered in [17], the Poisson bracket displays a certain rigidity with respect to the *uniform* norm of the functions. This rigidity is best expressed in terms of the *profile function* defined as follows [14].

Equip the space $\Pi := C^\infty(M) \times C^\infty(M)$ with the product uniform metric: $d((F, G), (H, K)) = \|F - H\| + \|G - K\|$. For each $s \geq 0$ define $\Pi_s := \{(H, K) \in \Pi \mid \|\{H, K\}\| \leq s\}$. In particular, Π_0 is the set of Poisson-commuting pairs. Given a pair $(F, G) \in \Pi$, define the *profile function* $\rho_{F,G} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$ by $\rho_{F,G}(s) = d((F, G), \Pi_s)$.

Question 4.18. Given a pair $(F, G) \in \Pi$, what can be said of $\rho_{F,G}(0)$? In other words, how well can (F, G) be approximated with respect to d by a Poisson-commuting pair?

Let us note that similar approximation questions have been extensively studied for matrices – see e.g. [37] and the references therein. It follows from [25] that the sets Π_s , $s \geq 0$, are closed with respect to d and therefore $\rho_{F,G}(s) > 0$ for $s \in [0, \|\{F, G\}\|)$. Symplectic quasi-states help to give a more precise answer in certain cases.

Theorem 4.19 ([14]). *Let $\zeta : C(M) \rightarrow \mathbb{R}$ be a symplectic quasi-state.*

a. Assume $X, Y, Z \subset M$ are closed sets that are superheavy with respect to ζ and satisfy $X \cap Y \cap Z = \emptyset$. Assume $F|_X \leq 0$, $G|_Y \leq 0$, $(F + G)|_Z \geq 1$ and at least one of the functions F, G has its range in $[0, 1]$. Then $\rho_{F,G}(0) = 1/2$ and for some positive constant C , independent of F, G , and for all $s \in [0; \|\{F, G\}\|]$

$$\frac{1}{2} - C\sqrt{s} \leq \rho_{F,G}(s) \leq \frac{1}{2} - \frac{s}{2\|\{F, G\}\|}.$$

b. Let $X_0, X_1, Y_0, Y_1 \subset M$ be closed sets so that $X_0 \cap X_1 = Y_0 \cap Y_1 = \emptyset$ and the sets $X_0 \cup Y_0, Y_0 \cup X_1, X_1 \cup Y_1, Y_1 \cup X_0$ are all superheavy with respect to ζ . Assume

$F, G \in C^\infty(M)$, $F|_{X_0} \leq 0$, $F|_{X_1} \geq 1$, $G|_{Y_0} \leq 0$, $G|_{Y_1} \geq 1$ and at least one of the functions F, G has its range in $[0, 1]$. Then $\rho_{F,G}(0) = 1/2$ and for some positive constant C , independent of F, G , and for all $s \in [0; ||\{F, G\}||]$

$$\frac{1}{2} - Cs \leq \rho_{F,G}(s) \leq \frac{1}{2} - \frac{s}{2||\{F, G\}||}.$$

The proof of part (b) uses the fact that $pb_4(X_0, Y_0, X_1, Y_1) > 0$ (see Remark 4.16) and part (a) is based on the positivity of a similar Poisson bracket invariant $pb_3(X, Y, Z)$ – see [14] for more details, as well as for examples where the theorem can be applied, including an example where the lower bound in part (a) is asymptotically sharp. For a version of Theorem 4.19 for *iterated* Poisson brackets see [27].

Here is another fact concerning the C^0 -rigidity of Poisson brackets whose proof uses partial symplectic quasi-states and the strong version of their partial quasi-additivity (see Theorem 3.2). Let $\mathcal{U} = \{U_1, \dots, U_N\}$ be a finite cover of M by displaceable open sets. Given a partition of unity $\vec{F} = \{F_1, \dots, F_N\}$ subordinated to \mathcal{U} (that is, $\text{supp } F_i \subset U_i$ for every i), consider the following measure of its Poisson non-commutativity:

$$\kappa(\vec{F}) := \inf_{x, y \in [-1, 1]^N} \left\| \left\{ \sum_{i=1}^N x_i F_i, \sum_{j=1}^N y_j F_j \right\} \right\|,$$

where the infimum is taken over all $x = (x_1, \dots, x_N), y = (y_1, \dots, y_N) \in [-1, 1]^N$. Set $pb(\mathcal{U}) := \inf_{\vec{F}} \kappa(\vec{F})$, where the infimum is taken over all partitions of unity \vec{F} subordinated to \mathcal{U} . We say that \mathcal{U} is *dominated by an open set* $U \subset M$ if for each $i = 1, \dots, N$ there exists $\phi_i \in \text{Ham}(M)$ so that $U_i \subset \phi_i(U)$.

Theorem 4.20 ([57]). *Assume \mathcal{U} is dominated by a displaceable open set U . Then there exists a constant $C = C(U) > 0$ so that*

$$pb(\mathcal{U}) \geq C/N^2. \quad (2)$$

The theorem strengthens a similar result proved previously in [28]. It is not clear whether the inequality (2) can be improved – see [57] for a discussion.

Remark 4.21. Amazingly, Theorem 4.20 that belongs to the mathematical formalism of classical mechanics can be used to prove results about mathematical objects of quantum nature appearing in the Berezin-Toeplitz quantization of a symplectic manifold – see [56, 57, 58].

4.4. Quasi-morphisms and metric properties of $\text{Ham}(M)$. The group $\text{Ham}(M)$ carries various interesting metrics. Here we will discuss how Calabi quasi-morphisms can be used to study these metrics.

The most remarkable metric on $\text{Ham}(M)$ is the Hofer metric. Namely, define the *Hofer norm* $||\phi||_H$ of $\phi \in \text{Ham}(M)$ as $||\phi||_H = \inf_F \int_0^1 (\max_M F_t - \min_M F_t) dt$, where the infimum is taken over all (time-dependent) Hamiltonians F generating

ϕ (if M is open, F is also required to be compactly supported). The *Hofer metric* is defined by $\varrho(\phi, \psi) = \|\phi\psi^{-1}\|_H$. It is a deep result of symplectic topology that ϱ is a bi-invariant metric – see e.g. [47] and the references therein.

Assume $\text{Ham}(M)$ admits a partial Calabi quasi-morphism μ . Then the stability property of μ (see Theorem 3.1) implies that μ is Lipschitz with respect to ϱ and therefore the diameter of $\text{Ham}(M)$ with respect to ϱ is infinite [20]. Moreover, the Lipschitz property of μ with respect to ϱ allows to obtain the following result on the growth of 1-parametric subgroups of $\text{Ham}(M)$ with respect to the Hofer norm.

Theorem 4.22 ([58], cf. [20]). *Assume $\text{Ham}(M)$ admits a partial Calabi quasi-morphism. Then there exists a set $\Xi \subset C^\infty(M)$ which is C^0 -open and C^∞ -dense in $C^\infty(M)$ so that $\lim_{t \rightarrow +\infty} \frac{\|\phi_F^t\|_H}{t\|F\|} > 0$ for any $F \in \Xi$.*

Calabi quasi-morphisms can be also applied to the study of the metric induced by ϱ on certain spaces of Lagrangian submanifolds of M – see [39, 63].

The group $\text{Ham}(M)$ also carries the C^0 -topology: equip M with a distance function d , given by a Riemannian metric on M , assume d is bounded, and define the C^0 -topology on $\text{Ham}(M)$ as the one induced by the metric $\text{dist}(\phi, \psi) = \max_{x \in M} d(\phi(x), \psi(x))$. The relation between the C^0 -topology and the Hofer metric on $\text{Ham}(M)$ is rather delicate (for instance, the C^0 -metric is never continuous with respect to the Hofer metric). One can use the Calabi quasi-morphisms on $\text{Ham}(B^{2n})$ (see Remark 3.11) in order to construct infinitely many linearly independent homogeneous quasi-morphisms on $\text{Ham}(B^{2n})$ that are both Lipschitz with respect to the Hofer metric and continuous in the C^0 -topology [26]. This yields the following corollary answering a question of Le Roux [43]:

Corollary 4.23 ([26]). *For any $c \in \mathbb{R}$ the set $\{\phi \in \text{Ham}(B^{2n}) \mid \|\phi\|_H \geq c\}$ has a non-empty interior in the C^0 -topology.*

See [62] for an extension of this result to a wider class of open symplectic manifolds.

(Partial) Calabi quasi-morphisms can be also applied to the study of the norms $\|\phi\|_U$ and $\|\phi\|_{U,0}$ on $\text{Ham}(M)$ (see Section 3.1) and the metrics defined by them. Namely, let $U \subset M$ be a displaceable open set. Assume $\text{Ham}(M)$ admits a Calabi quasi-morphism μ . A standard fact about quasi-morphisms bounded on a generating set yields that there exists a constant $C = C(\mu) > 0$ so that $\|\phi\|_{U,0} \geq C|\mu(\phi)|$ for any $\phi \in \text{Ham}(M)$. In particular, $\text{Ham}(M)$ is unbounded with respect to the norm $\|\cdot\|_{U,0}$.

On the other hand, if $\mu : \text{Ham}(M) \rightarrow \mathbb{R}$ is only a partial *but not genuine* Calabi quasi-morphism, it can be used to show the unboundedness of $\text{Ham}(M)$ with respect to the norm $\|\cdot\|_U$ [15]. Namely, assume $X, Y = \varphi(X) \subset M$, $\varphi \in \text{Symp}_0(M)$, are heavy¹⁴ disjoint¹⁵ closed subsets of M and V, W are their disjoint

¹⁴With respect to the partial symplectic quasi-state ζ associated to μ . If X is heavy with respect to ζ , then so is $\varphi(X)$, since ζ is $\text{Symp}_0(M)$ -invariant.

¹⁵This is possible only if μ is *not* a genuine Calabi quasi-morphism, since otherwise each heavy set (with respect to ζ) is also superheavy and therefore must intersect any other heavy set.

open neighborhoods (for instance, X and Y can be two meridians on a standard symplectic torus). Let F, G be smooth functions supported, respectively, in V, W so that $F|_X \equiv 1 = \max_M F$, $G|_Y \equiv 1 = \max_M G$. One can easily show that

$$k = |\mu(\phi_F^k \phi_G^k) - \mu(\phi_F^k) - \mu(\phi_G^k)| \leq C \|\phi_F^k\|_U$$

for any $k \in \mathbb{N}$ and a constant $C > 0$ depending only on μ and U . Thus in such a case $\|\phi_F^k\|_U$ grows asymptotically linearly with k and the norm $\|\cdot\|_U$ is unbounded on $\text{Ham}(M)$.

Question 4.24. Does there exist a closed symplectic manifold M for which $\|\cdot\|_U$ is bounded on $\text{Ham}(M)$?

4.5. First steps of symplectic function theory – discussion. A smooth manifold M and various geometric structures on it can be described in terms of the function space $C^\infty(M)$ – for instance, subsets of M correspond to ideals in $C^\infty(M)$, tangent vectors to derivations on $C^\infty(M)$ etc. In particular, a symplectic structure on M is completely determined by the corresponding Poisson brackets on $C^\infty(M)$ which means that, in principle, any symplectic phenomenon has a counterpart in the *symplectic function theory*, that is, the function theory of the Poisson brackets. The key feature of symplectic topology is C^0 -rigidity appearing in various forms for smooth objects on symplectic manifolds. Its counterpart in the symplectic function theory is the rigidity of the Poisson brackets with respect to the C^0 -norm on functions – see e.g. [58] for a deduction of the foundational Eliashberg-Gromov theorem on the C^0 -closedness of $\text{Symp}(M)$ from the C^0 -rigidity of the Poisson brackets.

The results and methods presented in this survey show that thinking about symplectic phenomena in terms of the function theory may have a number of advantages. First, it allows to use the “Lie group - Lie algebra” connection between $\widehat{\text{Ham}}(M)$ and $C^\infty(M)/\mathbb{R}$: most of the properties of symplectic quasi-states are proved using the properties of Calabi quasi-morphisms. Second, it allows to deal with singular sets (see Remark 4.6). Third, it allows to apply functional methods, like averaging, to Hamiltonian dynamics (see Remark 4.16). Fourth, it helps to find connections between symplectic topology and quantum mechanics since it is the Poisson algebra $C^\infty(M)$ that is being quantized in various quantization constructions (see e.g. Remark 4.21). Moreover, one may hope to discover new geometric and dynamical phenomena by studying the function theory of the Poisson bracket. For instance, the behavior of the profile function $\rho(t)$ as $t \rightarrow 0$ (see Section 4.3 and [14]) is obviously of interest in symplectic function theory but its geometric or dynamical implications are absolutely unclear.

References

- [1] Aarnes, J.F., Quasi-states and quasi-measures, *Adv. Math.* **86** (1991), 41-67.
- [2] Aarnes, J.F., Construction of non-subadditive measures and discretization of Borel measures, *Fund. Math.* **147** (1995), 213-237.

- [3] Albers, P., On the extrinsic topology of Lagrangian submanifolds, *Int. Math. Res. Not.* **38** (2005), 2341–2371. Erratum: *Int. Math. Res. Not.* (2010), 1363–1369.
- [4] Alston, G., Lagrangian Floer homology of the Clifford torus and real projective space in odd dimensions, *J. Sympl. Geom.* **9** (2011), 83–106.
- [5] Banyaga, A., Sur la structure du groupe des difféomorphismes qui préservent une forme symplectique, *Comm. Math. Helv.* **53** (1978), 174–227.
- [6] Bell, J.S., On the problem of hidden variables in quantum mechanics, *Rev. Modern Phys.* **38** (1966), 447–452.
- [7] Ben Simon, G., The nonlinear Maslov index and the Calabi homomorphism, *Comm. Contemp. Math.* **9** (2007), 769–780.
- [8] Biran, P., Cornea, O., Rigidity and uniruling for Lagrangian submanifolds, *Geom. Topol.* **13** (2009), 2881–2989.
- [9] Biran, P., Entov, M., Polterovich, L., Calabi quasimorphisms for the symplectic ball, *Comm. Contemp. Math.* **6** (2004), 793–802.
- [10] Borman, M.S., Symplectic reduction of quasi-morphisms and quasi-states, *J. Sympl. Geom.* **10** (2012), 225–246.
- [11] Borman, M.S., Quasi-states, quasi-morphisms, and the moment map, *Int. Math. Res. Not.* (2013), 2497–2533.
- [12] Borman, M.S., Zapolsky, F., Quasi-morphisms on contactomorphism groups and contact rigidity, preprint, arXiv:1308.3224, 2013.
- [13] Branson, M., Symplectic manifolds with vanishing action-Maslov homomorphism, *Algebr. Geom. Topol.* **11** (2011), 1077–1096.
- [14] Buhovsky, L., Entov, M., Polterovich, L., Poisson brackets and symplectic invariants, *Selecta Math. (N.S.)* **18** (2012), 89–157.
- [15] Burago, D., Ivanov, S., Polterovich, L., Conjugation-invariant norms on groups of geometric origin, in *Groups of Diffeomorphisms: In Honor of Shigeyuki Morita on the Occasion of His 60th Birthday*. Adv. Studies in Pure Math. 52, Math. Soc. of Japan, Tokyo, 2008.
- [16] Calegari, D., *scl*. MSJ Memoirs 20, Math. Soc. of Japan, Tokyo, 2009.
- [17] Cardin, F., Viterbo, C., Commuting Hamiltonians and Hamilton-Jacobi multi-time equations, *Duke Math. J.* **144** (2008), 235–284.
- [18] Eliashberg, Y., Polterovich, L., Symplectic quasi-states on the quadric surface and Lagrangian submanifolds, preprint, arXiv:1006.2501, 2010.
- [19] Entov, M., Commutator length of symplectomorphisms, *Comm. Math. Helv.* **79** (2004), 58–104.
- [20] Entov, M., Polterovich, L., Calabi quasimorphism and quantum homology, *Int. Math. Res. Not.* **30** (2003), 1635–1676.
- [21] Entov, M., Polterovich, L., Quasi-states and symplectic intersections, *Comm. Math. Helv.* **81** (2006), 75–99.
- [22] Entov, M., Polterovich, L., Symplectic quasi-states and semi-simplicity of quantum homology, in *Toric Topology* (eds. M.Harada, Y.Karshon, M.Masuda and T.Panov), 47–70. Contemp. Math. 460, AMS, Providence RI, 2008.

- [23] Entov, M., Polterovich L., Rigid subsets of symplectic manifolds, *Compositio Math.* **145** (2009), 773–826.
- [24] Entov, M., Polterovich, L., Lie quasi-states, *J. Lie Theory* **19** (2009), 613–637.
- [25] Entov, M., Polterovich, L., C^0 -rigidity of Poisson brackets, in *Proceedings of the Joint Summer Research Conference on Symplectic Topology and Measure-Preserving Dynamical Systems* (eds. A. Fathi, Y.-G. Oh and C. Viterbo), 25–32. Contemp. Math. 512, AMS, Providence RI, 2010.
- [26] Entov, M., Polterovich, L., Py, P., On continuity of quasimorphisms for symplectic maps. With an appendix by Michael Khanovsky, in *Perspectives in analysis, geometry, and topology*, 169–197. Progr. Math. 296, Birkhuser/Springer, New York, 2012.
- [27] Entov, M., Polterovich L., Rosen, D., Poisson brackets, quasi-states and symplectic integrators, *Discr. and Cont. Dyn. Systems* **28** (2010), 1455–1468.
- [28] Entov, M., Polterovich, L., Zapolsky, F., Quasi-morphisms and the Poisson bracket, *Pure and Appl. Math. Quarterly* **3** (2007), 1037–1055.
- [29] Entov, M., Polterovich, L., Zapolsky, F., An anti-Gleason phenomenon and simultaneous measurements in classical mechanics, *Found. of Physics* **37** (2007), 1306–1316.
- [30] Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K., *Lagrangian intersection Floer theory: anomaly and obstruction. Parts I, II*. AMS, Providence, RI, International Press, Somerville, MA, 2009.
- [31] Fukaya, K., Oh, Y.-G., Ohta, H., Ono, K., Spectral invariants with bulk, quasimorphisms and Lagrangian Floer theory, arXiv:1105.5123, 2011.
- [32] Galkin, S., The conifold point, preprint, arXiv:1404.7388, 2014.
- [33] Gambaudo, J.-M., Ghys, E., Commutators and diffeomorphisms of surfaces, *Erg. Th. Dyn. Sys.* **24** (2004), 1591–1617.
- [34] Givental, A., The nonlinear Maslov index, in *Geometry of low-dimensional manifolds, 2 (Durham, 1989)*, 35–43. London Math. Soc. Lecture Note Ser. 151, Cambridge Univ. Press, Cambridge, 1990.
- [35] Givental, A., Nonlinear generalization of the Maslov index, in *Theory of singularities and its applications*, 71–103. Adv. Soviet Math. 1, AMS, Providence, RI, 1990.
- [36] Gleason, A.M., Measures on the closed subspaces of a Hilbert space, *J. Math. Mech.* **6** (1957), 885–893.
- [37] Hastings, M.B., Making almost commuting matrices commute, *Comm. Math. Phys.* **291** (2009), 321–345.
- [38] Kawasaki, M., Superheavy subsets and noncontractible Hamiltonian circle actions, preprint, 2013.
- [39] Khanovsky, M., Hofer’s metric on the space of diameters, *J. Topol. and Analysis* **1** (2009), 407–416.
- [40] Kislev, A., Compactly supported Hamiltonian loops with non-zero Calabi invariant, preprint, arXiv:1310.1555, 2013.
- [41] Knudsen, F.F., Topology and the construction of extreme quasi-measures, *Adv. Math.* **120** (1996), 302–321.
- [42] Lanzat, S., Quasi-morphisms and symplectic quasi-states for convex symplectic manifolds, *Int. Math. Res. Not.* (2013), 5321–5365.

- [43] Le Roux, F., Six questions, a proposition and two pictures on Hofer distance for Hamiltonian diffeomorphisms on surfaces, in *Symplectic topology and measure preserving dynamical systems*, 33–40. Contemp. Math. 512, AMS, Providence, RI, 2010.
- [44] Maydanskiy, M., Mirabelli, B.P., Semisimplicity of the quantum cohomology for smooth Fano toric varieties associated with facet symmetric polytopes, *Electron. Res. Announc. Math. Sci.* **18** (2011), 131–143.
- [45] McDuff, D., Monodromy in Hamiltonian Floer theory, *Comment. Math. Helv.* **85** (2010), 95–133.
- [46] McDuff, D., Displacing Lagrangian toric fibers via probes, in *Low-dimensional and symplectic topology*, 131–160. Proc. Sympos. Pure Math. 82, AMS, Providence, RI, 2011.
- [47] McDuff, D., Salamon, D., *Introduction to symplectic topology*. Second edition. Oxford Univ. Press, New York, 1998.
- [48] McDuff, D., Salamon, D., *J-holomorphic curves and symplectic topology*. Second edition. AMS Colloquium Publ. 52, AMS, Providence, RI, 2012.
- [49] Monzner, A., Vichery, N., Zapolsky, F., Partial quasimorphisms and quasistates on cotangent bundles, and symplectic homogenization, *J. Mod. Dyn.* **6** (2012), 205–249.
- [50] Oakley, J., Usher, M., On certain Lagrangian submanifolds of $S^2 \times S^2$ and $\mathbb{C}P^n$, preprint, arXiv:1311.5152, 2013.
- [51] Oh, Y.-G., Symplectic topology as the geometry of action functional I, *J. Diff. Geom.* **46** (1997), 499–577.
- [52] Oh, Y.-G., Symplectic topology as the geometry of action functional II, *Comm. Analysis Geom.* **7** (1999), 1–55.
- [53] Oh, Y.-G., Construction of spectral invariants of Hamiltonian paths on general symplectic manifolds, in *The breadth of symplectic and Poisson geometry*, 525–570. Birkhäuser, Boston, 2005.
- [54] Ostrover, Y., Calabi quasi-morphisms for some non-monotone symplectic manifolds, *Algebr. Geom. Topol.* **6** (2006), 405–434.
- [55] Ostrover, Y., Tyomkin, I., On the quantum homology algebra of toric Fano manifolds, *Selecta Math. (N.S.)* **15** (2009), 121–149.
- [56] Polterovich, L., Quantum unsharpness and symplectic rigidity, *Lett. Math. Phys.* **102** (2012), 245–264.
- [57] Polterovich, L., Symplectic geometry of quantum noise, preprint, arXiv:1206.3707, 2012. To appear in *Comm. Math. Phys.*
- [58] Polterovich, L., Rosen., D., *Function theory on symplectic manifolds*, book draft, 2014.
- [59] Py, P., Quasi-morphismes et invariant de Calabi, *Ann. Sci. École Norm. Sup. (4)* **39** (2006), 177–195.
- [60] Py, P., Quasi-morphismes de Calabi et graphe de Reeb sur le tore, *C. R. Math. Acad. Sci. Paris* **343** (2006), 323–328.
- [61] Schwarz, M., On the action spectrum for closed symplectically aspherical manifolds, *Pacific J. Math.* **193** (2000), 419–461.
- [62] Seyfaddini, S., Descent and C^0 -rigidity of spectral invariants on monotone symplectic manifolds, *J. Topol. Analysis* **4** (2012), 481–498.

- [63] Seyfaddini, S., Unboundedness of the Lagrangian Hofer distance in the Euclidean ball, preprint, arXiv:1310.1057, 2013.
- [64] Shelukhin, E., Remarks on invariants of Hamiltonian loops, *J. Topol. Analysis* **2** (2010), 277–325.
- [65] Shelukhin, E., The action homomorphism, quasimorphisms and moment maps on the space of compatible almost complex structures, preprint, arXiv:1105.5814, 2011.
- [66] Shtern, A.I., The Kazhdan-Milman problem for semisimple compact Lie groups, *Russian Math. Surveys* **62** (2007), 113–174.
- [67] Tamarkin, D., Microlocal condition for non-displaceability, preprint, arXiv:0809.1584, 2008.
- [68] Usher, M., Spectral numbers in Floer theories, *Compositio Math.* **144** (2008), 1581–1592.
- [69] Usher, M., Duality in filtered Floer-Novikov complexes, *J. Topol. Analysis* **2** (2010), 233–258.
- [70] Usher, M., Deformed Hamiltonian Floer theory, capacity estimates and Calabi quasimorphisms, *Geom. Topol.* **15** (2011), 1313–1417.
- [71] Viterbo, C., Symplectic topology as the geometry of generating functions, *Math. Ann.* **292** (1992), 685–710.
- [72] Von Neumann, J., *Mathematical foundations of quantum mechanics*. Princeton University Press, Princeton, 1955. (Translation of *Mathematische Grundlagen der Quantenmechanik*. Springer, Berlin, 1932.)
- [73] Wu, W., On an exotic Lagrangian torus in $\mathbb{C}P^2$, preprint, arXiv:1201.2446, 2012.

Department of Mathematics
 Technion – Israel Institute of Technology
 Haifa 32000, Israel
 E-mail: entov@math.technion.ac.il